AN ANALYTICAL MODEL FOR POLLUTANTS DISPERSION RELEASED FROM DIFFERENT SOURCES IN ATMOSPHERIC BOUNDARY LAYER

Kumar Anikender and Goyal P.*
Centre for Atmospheric Sciences, Indian Institute of Technology Delhi, Hauz Khas, New Delhi (INDIA)

Received July 05, 2012
Accepted September 20, 2012

ABSTRACT

An analytical model for dispersion of air pollutants released from point, line and area sources is formulated by considering the wind speed as a power law profile of vertical height above the ground and vertical eddy diffusivity as an explicit function of downwind distance from the source and vertical height in Neumann (total reflection) boundary conditions. The standard deviation of eddy diffusivity in lateral direction depends on the downwind distance. The analytical formulation of the line source is obtained by integrating the formulation of point source in crosswind direction and area source is obtained by integrating the point source in downwind and crosswind directions. An emission inventory of Respirable Suspended Particulate Mater (RSPM) emitted from different sources namely vehicles, domestic, industries and power plants, has been made using the primary and secondary data in Delhi. The analytical model is evaluated using observed concentration at various locations in Delhi obtained from Central Pollution Control Board (CPCB), Delhi, India and National Environment Engineering Research Institute (NEERI), Nagpur, India. The model’s predicted values are found well in agreement with the observed values. The performance of model is found to be satisfactory. The same model can also be used for any urban city.

Key Words: Analytical model, Point source, RSPM, Eddy diffusivity, Emission inventory

INTRODUCTION

The atmospheric dispersion equation has long been used to describe the transport of air borne pollutants in a turbulent atmosphere. The use of analytical solutions of this equation was the first and remains the convenient way for modeling air pollution. Air dispersion models based on its analytical solutions posses several advantages over numerical models, because all the influencing parameters are explicitly expressed in a mathematically closed form. Analytical models are indispensable tools to study specific problems associated with the adverse effects of the air pollutants in the atmosphere as they provide the direct insight into the parameters controlling dispersion and related physics with a simple and fast flexible way.

Several efforts have been made to obtained analytical solution of the advection diffusion equation for point, line and area sources, which are usually based on assumptions of wind speed and vertical eddy diffusivity both as either a power function of height or as a function of downwind distance. Has solved the advection diffusion equation analytically for area source with wind speed and vertical eddy diffusivity both as power function of vertical height with unbounded region. In general, the eddy diffusivity should be a function of both vertical height above the ground and downwind distance from the source. In view of the above discussion, the present study is aimed to develop an analytical model for dispersion of air pollutants released from point, line and area sources by considering the wind speed as a power law profile of vertical height above the ground and vertical eddy diffusivity as an explicit function of downwind distance from the source and vertical height for Neumann (total reflection) boundary conditions in atmospheric boundary layer i.e., bounded by boundary layer or mixing/inversion height. The lowest region of the atmosphere, which experiences surface
effects through vertical exchanges of heat, momentum and moisture, is called the Atmospheric Boundary Layer (ABL). Since most of the industrial stacks, vehicular and domestic sources are located within this layer; it plays a pivotal role in the dispersion of air pollutants because the pollutants emitted near the surface are largely diluted and confined within it.

The mathematical description of the models has been discussed in section 2. The methodology is given in section 3. The results and discussions are given in section 4. The section 5 summarizes the conclusions.

**Analytical solutions of the Advection diffusion equation**

The steady state transport of a non reactive contaminant released from an elevated point source can be described by

\[
U(z) \frac{\partial C}{\partial x} = \frac{\partial}{\partial y} \left( K_y(x, z) \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_z(x, z) \frac{\partial C}{\partial z} \right)
\]

(1)

where \(x, y, z\) are coordinates in the along-wind, crosswind, and vertical directions respectively. \(C\) is the mean concentration of pollutants, and \(U(z)\) is the mean wind speed in downwind direction. \(K_y(x, z)\) and \(K_z(x, z)\) are eddy diffusivities of pollutants in the crosswind and vertical direction respectively. Transport by turbulent diffusion in the along wind direction is neglected compared to the advection in the along wind direction. Eq. (1) is solved with the following boundary conditions:

(i) The ground is assumed to be a perfectly reflecting surface and accordingly diffusive flux vanishes close to the surface, i.e.

\[
K_z(x, z) \frac{\partial C}{\partial z} \rightarrow 0 \quad \text{at } z \rightarrow 0 \\
C \rightarrow 0 \quad \text{as } y \rightarrow \pm \infty
\]

(ii) The pollutant can not penetrate the top of the inversion/mixed layer

\[
K_z(x, z) \frac{\partial C}{\partial z} \rightarrow 0 \quad \text{at } z \rightarrow h
\]

(iii) The pollutant is released from an elevated point source of strength \(Q_p\) located at the point \((0, y, z_0)\),

\[
U(z)C(0, y, z) = Q_p \delta(y - y_0) \delta(z - z_0)
\]

(4)

where \(\delta\) is the Dirac delta function and point source is located at \((0, y_0, z_0)\).

The transport of contaminant primarily depends on the wind speed \(U\). The formulations of the commonly used dispersion models in air quality studies assume wind speed to be constant. However, it is well known that wind speed increases with height in the lower part of the atmospheric boundary layer.\(7,8\) The height dependent wind speed can be expressed as

\[
U(z) = a x^\alpha, \ a = U(z_r) z_r^{-\alpha}
\]

(5)

where \(U(z_r)\) is the wind speed at reference height \(z_r\) and \(\alpha\) depends on atmospheric stability.

In dispersion models, \(K_z\) is parameterized as a function of the height \(z\) above the ground.\(4-5\) However, based on the Taylor’s analysis and statistical theory, the eddy diffusivity depends on the downwind distance \(x\) also\(9\). Therefore, the eddy diffusivity is considered as a function of \(x\) and \(z\) both\(6,9\). Hence, the modified form of \(K_z(x, z)\) is given as:

\[
K_z(x, z) = K_z'(z) f(x)
\]

(6)

where \(K_z'(z)\) is parameterized as a power law profile in \(z\):

\[
K_z'(z) = b z^\beta, \ b = K_z'(z_r) z_r^{-\beta}
\]

(7)

where \(K_z'(z_r)\) is the value of \(K_z'\) at height \(z = z_r\) and \(\beta\) depends on atmospheric stability.

Using Taylor’s hypothesis, the lateral eddy diffusivity can be represented by\(10-12\)

\[
K_y(x, z) = \frac{1}{2} U(z) \frac{d \sigma_y^2(x)}{dx}
\]

(8)

where \(\sigma_y\) is the coefficient of diffusivity in the crosswind direction.

The analytical solution of point source using Eq. (1) for the profiles of wind speed (Eq. (5) and eddy diffusivities (Eq.6) and (Eq.8), with boundary conditions Eq. 2 to Eq. 4 is obtained as:

\[
C(x, y, z) = \frac{Q_p}{\sqrt{2 \pi \sigma_y}} \left[ \frac{\alpha + 1}{ah^{a+1}} + \frac{\alpha - \beta + 2}{ah^{a+\beta}} (zh)^{(a-\beta)/2} \times \right.
\]

\[
\sum_{n=1}^{\infty} J_n \left[ \gamma_n(z/h)^{(a-\beta)/2} J_n(z_0/h)^{(a-\beta)/2} \right] \times
\]

\[
\exp \left[ -\frac{b(\alpha - \beta + 2)^2 y^2}{4ah^{a-\beta+2}} \int_0^x f(x')dx' \right] \times
\]

\[
\exp \left[ -\frac{(y - y_0)^2}{2\sigma_y^2} \right]
\]

(9)
where \( \mu = (1 - \beta) / (\alpha - \beta + 2) \), \( J_{\mu+1} \) is the Bessel function of order \(-\mu\) and \( \gamma_n \)'s are zeros as shown in the following equation:

\[
J_{\mu+1}(\gamma_n) = 0
\]

(10), and \( f(x) \) is expressed as a linear function of downwind distance as:

\[
f(x) = \gamma U x.
\]

The details of Eq (9) have been given in Appendix A.

**Line source model**

For the infinite line source with strength \( Q_n \), the solution can be obtained by integrating Eq. (9) from \( y_s = -\infty \) to \( \infty \).

\[
C(x, z) = 0 = \int_{y_s}^{y_e} Q_n \times \left[ \frac{\alpha + 1}{Ah^{\alpha + 1}} + \frac{\alpha - \beta + 2}{Ah^{\alpha + 2}} (zz) \right]^{(\beta - 1)/2}
\]

\[
\times \sum_{n=1}^{\infty} J_{\mu}(y_n) \frac{\alpha - \beta + 2}{Ah^{\alpha + 2}} \left[ J_{\mu}(y_n) \right]^{(\beta - 1)/2}\]

\[
\times \exp \left( \frac{-b(\alpha - \beta + 2)^2 y_n^2}{4ah^{\alpha + 2}} \right) \int_0^\infty f(x')dx'
\]

(11)

For the finite line source of strength \( Q_n \), the solution can be obtained by integrating Eq. (9) from \( y_s = y_i \) to \( y_e \).

\[
C(x, y, z) = \int_{y_i}^{y_e} C(x - x_s, y, z)dx_s
\]

(13)

where \( f(x) \) is a linear function of downwind distance as:

\[
f(x) = \gamma U x.
\]

\[
C(x, z) = \int_{x_s}^{x_e} Q_n \times \left[ \frac{\alpha + 1}{Ah^{\alpha + 1}} + \frac{\alpha - \beta + 2}{Ah^{\alpha + 2}} (zz) \right]^{(\beta - 1)/2}
\]

\[
\times \sum_{n=1}^{\infty} J_{\mu}(y_n) \frac{\alpha - \beta + 2}{Ah^{\alpha + 2}} \left[ J_{\mu}(y_n) \right]^{(\beta - 1)/2}\]

\[
\times \exp \left( \frac{-b(\alpha - \beta + 2)^2 y_n^2}{4ah^{\alpha + 2}} \right) \int_0^\infty f(x')dx'
\]

(11)

For the finite area source of strength \( Q_n \), the solution is obtained by integrating Eq. (12) from \( x_s = x_i \) to \( x_e \).

\[
C(x, y, z) = \int_{x_i}^{x_e} C(x - x_s, y, z)dx_s
\]

(15)

where \( \sigma_y = Rx^{-1} \), where \( \overline{U} \) is average velocity, \( \gamma \) is turbulence parameter and \( R \) and \( r \) are constants depending on the atmospheric stability. The turbulence parameter \( \gamma \) is parameterized as the square of turbulent intensity using Taylor statistical theory of diffusion:\n
\[
\gamma = \left( \frac{\sigma_w}{U} \right)^2.
\]

Turbulent intensity can be expressed as:

\[
\left( \frac{\sigma_w}{U} \right) = \tan(\sigma_\theta), \text{ where } \sigma_\theta.
\]

is the standard deviation of vertical wind direction in radians. For small, the turbulent intensity is approximated as:\n
After substituting above variables in Eq (15), the solution is obtained as:

\[
C(x, y, z) = \int_{x_i}^{x_e} Q_n \times \left[ \frac{\alpha + 1}{Ah^{\alpha + 1}} + \frac{\alpha - \beta + 2}{Ah^{\alpha + 2}} (zz) \right]^{(\beta - 1)/2}
\]

\[
\times \sum_{n=1}^{\infty} J_{\mu}(y_n) \frac{\alpha - \beta + 2}{Ah^{\alpha + 2}} \left[ J_{\mu}(y_n) \right]^{(\beta - 1)/2}\]

\[
\times \exp \left( \frac{-b(\alpha - \beta + 2)^2 y_n^2}{4ah^{\alpha + 2}} \right) \int_0^\infty f(x')dx'
\]

(11)

which is more general solution for finite area source and can be solved by numerical integration.
The concentration for a ground level source is obtained by taking the limit $z \to 0$ in Eq. (17) and using the property of Bessel function for small arguments:\(^14\)

$$J_1(\eta) \rightarrow \frac{\eta^\nu}{2^\nu \Gamma(1+\nu)} \text{as} \ \eta \to 0 \quad (18)$$

$$C(x, y, z) = \int_0^\infty \frac{Q_n}{2} \left[ \frac{\alpha + 1}{a h^{n+1}} + \frac{2^n (\alpha - \beta + 2)}{a \Gamma(1-\mu) R^{(2n-\beta+1)/2}} (z)^{(1-\beta)/2} \right]$$

$$\times \sum_{\alpha=1}^\infty \gamma_n^2 J_\alpha(y_n) \times \exp\left(-\frac{b(\alpha - \beta + 2)^2 y_n^2 Q(x-x_n)^2}{8ah^{n+1}} \right)$$

$$\times \left[ \text{erf}\left(\frac{y-y_1}{\sqrt{2R(x-x_n)}}\right) - \text{erf}\left(\frac{y-y_2}{\sqrt{2R(x-x_n)}}\right) \right] dx_n \quad (19)$$

where $\Gamma$ is the complete gamma function defined by

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$$

The concentration for a ground level source is obtained by taking the limit $z \to 0$ in Eq. (19) and using the property of Bessel function for small arguments,

$$C(x, y, 0) = \int_0^\infty \frac{Q_n}{2} \left[ \frac{\alpha + 1}{a h^{n+1}} + \frac{2^n (\alpha - \beta + 2)}{a \Gamma(1-\mu) R^{(2n-\beta+1)/2}} \right]$$

$$\times \sum_{\alpha=1}^\infty \gamma_n^2 J_\alpha(y_n) \times \exp\left(-\frac{b(\alpha - \beta + 2)^2 y_n^2 Q(x-x_n)^2}{8ah^{n+1}} \right)$$

$$\times \left[ \text{erf}\left(\frac{y-y_1}{\sqrt{2R(x-x_n)}}\right) - \text{erf}\left(\frac{y-y_2}{\sqrt{2R(x-x_n)}}\right) \right] dx_n \quad (20)$$

In these solutions $\alpha$ has relationship with Pasquill’s Stability Classes\(^14\) and the exponent $\beta = 1 - \alpha$ is based on the Schmidt’s conjugate law.

**MATERIAL AND METHODS**

In order to evaluate the performance of models presented in Section 2, a case study of Delhi has been made. The above formulations have been used to predict the concentration of RSPM due to point, line and area sources for the month of January 2008, representative of winter season.

First of all, a gridded emission inventory of RSPM has been developed over an area of 26 Km x 30 km of Delhi. The total area has been divided into 195 square grids of size 2 Km x 2 Km. Emission of RSPM has been estimated in each grid due to all anthropogenic sources viz., domestic, industries, power plants and vehicles for the year 2006 using the primary and secondary data. The gridded emission inventory (Fig.1) shows the spatial distribution of emissions of RSPM due to all types of sources. The emission from domestic sources has been distributed uniformly in all the grids and the emission from industries and traffic intersections have been apportioned in those grids where the industries and traffic intersections are located. In addition to this the emissions of RSPM due to three power plants namely Indraprastha, Badarpur and Rajghat are superimposed in their respective grids. The Fig.1 also reflects that grids of power plants in addition to traffic intersections have higher emissions compared to others. This emission inventory is used as input to the models.

From an atmospheric pollution perspective, the most important season in Delhi is the winter lasting from December to February. This period is dominating by cold, dry air and ground based inversion with low wind conditions ($u<1\text{ms}^{-1}$), which increase the concentration of pollutants.\(^16\) For practical reasons, the January month of 2008 is used in this case study. The hourly meteorological data, measured at Safdarjung Airport from India Meteorological Department (IMD) is used as second input to the models.

Atmospheric stability measurements, based on surface wind speed, cloud cover, time of day and solar insolation (strong, moderate, slight), were classified according to Pasquill’s stability classes of A, B, C, D, E and F, which range from extremely unstable to extremely stable as given by.\(^17\) The dispersion parameters have also been calculated based on the stability parameters.

**RESULTS AND DISCUSSION**

The 24 hourly averaged predicted concentration of RSPM for Jan 2008 has been obtained from the models by using the emission inventory and meteorological data as input parameters. The spatial distribution of RSPM concentration, obtained from models is shown in the form of isopleths in Fig. 2, which indicates the hot spots of RSPM at I.T.O., Nizamuddin, Badarpur power station, I.P. powers station, Rajghat power station, AIIMS, Sirifort etc., ranging from 300-700 $\mu\text{g/m}^3$. This Fig. 2 also shows the maximum concentration near the NH-24, which is justifiable due to the presence of maximum number of vehicles passing through and Shadhar industrial area in close proximity. The
Fig. 1: Emission inventory of RSPM (g/s) over Delhi city from all sources.

Table 1: Comparison of 24 hourly averaged predicted and observed concentration of RSPM at different locations in Delhi in Jan 2008.

<table>
<thead>
<tr>
<th>Location</th>
<th>Model predicted (µg/m³)</th>
<th>Observed (µg/m³)</th>
<th>National ambient air quality standard (µg/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pitampura</td>
<td>331.91</td>
<td>276.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Sirifort</td>
<td>473.92</td>
<td>354.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Jankpuri</td>
<td>320.23</td>
<td>387.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Shahzada Bagh</td>
<td>363.77</td>
<td>377.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Sarojini Nagar</td>
<td>447.92</td>
<td>486.00</td>
<td>100.00</td>
</tr>
</tbody>
</table>
Fig.2 : 24 hourly averaged concentration of RSPM (µg/m³) due to all types of sources in Jan 2008.

Mean Squared Error (NMSE) as 0.00029 (ideal value 0), correlation coefficient as 0.4557 (ideal value 1). The fractional bias is -0.03025, which reflects the over prediction of the models and are within a factor of two, which proves that the performance of models are satisfactory according to assessment of.\(^{18}\) This table also reveals that the concentration levels are always higher than the National Ambient Air Quality Standards (NAAQS).

**CONCLUSION**

In this study an analytical model for dispersion of air pollutants released from point, line and area sources is formulated by considering the wind speed as a power law profile of vertical height above the ground and vertical eddy diffusivity as an explicit function of downwind distance from the source and vertical height in Neumann (total...
reflection) boundary condition. These models have been used to predict 24 hourly concentration of RSPM in the month of Jan 2008 in Delhi. In this study an emission inventory of RSPM emitted from different types of primary and secondary sources namely vehicular, domestic, industries and power plants has been made in Delhi. The analytical models are evaluated with observed concentration at different locations in Delhi obtained from CPCB, Delhi, India and NEERI, Nagpur, India which show that the models lead the slightly over prediction with the observation and 100% values are within a factor of two. The statistical error analysis shows that the model’s predicted values are well in agreement with the observed values. The results also show that the concentration levels are always high in comparison to the NAAQS. Although the present models have the limitation as the longitudinal diffusion is neglected in comparison to the advection and the models are not treating the wind directions at different vertical heights.

**Appendix A**

Based on the analysis of Prairie Grass and some other historical tracer experiments of atmospheric dispersion\(^9\) concluded that the ground level crosswind concentration profile of dispersing plume on average is well characterized as having a Gaussian shape, which is well predicted by all atmospheric transport and diffusion models, regardless of their sophistication. Thus, by assuming the Gaussian concentration distribution in crosswind direction,\(^9\)\(^,\)\(^19\)\(^,\)\(^20\) the steady state concentration of a pollutant released from point source in a three dimensional domain can be described as

\[
C(x, y, z) = C(x, z) \frac{\exp(-y^2 / 2\sigma_y^2)}{\sqrt{2\pi} \sigma_y}, \quad (A1),
\]

where \(C(x, z)\) is the crosswind integrated concentration and can be obtained using the separation of variable technique in total reflection boundary condition.

**REFERENCES**


